THE BOUNDARY VALUE PROBLEMS AND SEMI-GROUPS ASSOCIATED WITH CERTAIN INTEGRO-DIFFERENTIAL OPERATORS

BY

JOANNE ELLIOTT(1)

Introduction. The object of this paper is to study initial value problems for the equations

$$(0.1) \qquad u_t(x, t) = \frac{1}{\pi} \overline{P} \int_{-1}^{+1} \frac{u_\xi(\xi, t)d\xi}{\xi - x} = \Omega u$$

and

$$(0.2) \qquad v_t(x, t) = \frac{1}{\pi} \frac{d}{dx} \overline{P} \int_{-1}^{+1} \frac{v(\xi, t)d\xi}{\xi - x} = \Omega^* v,$$

where the integrals are taken in the sense of the extended principal value of Definition 1.1. This problem, suggested by W. Feller, arises in the theory of stochastic processes. However, it is interesting from a purely analytic point of view, and this paper will be concerned only with that aspect. A probabilistic interpretation of our result will be found in a forthcoming paper by Feller. For a precise definition of the operators Ω and Ω^* see §1.

The equation

$$(0.3) \qquad u_t(x, t) = \pi^{-1} P \int_{-\infty}^{+\infty} \frac{u_\xi(\xi, t)d\xi}{\xi - x} = -I^{-1}u,$$

where the underlying space is $C[-\infty, +\infty]$, has been considered as the backward equation for the Cauchy process in [2]. The solutions are harmonic functions in the upper half-plane. We may consider the solutions of (0.3) as transformations on the initial values, that is for $f \in C[-\infty, +\infty]$, we define $T_t f = u(t, x)$, where u is the solution of (0.3) such that $u(0, x) = f(x)$. These transformations have the semi-group property $T_{t+h}f = T_t T_h f$. In view of this, we might expect the semi-group given by

$$(0.4) \qquad T_t f = \pi^{-1} \int_{-\infty}^{+\infty} \frac{f(\xi)t d\xi}{(\xi - x)^2 + t^2},$$

where $f \in C[-\infty, +\infty]$, to have $-I^{-1}$ as infinitesimal generator. This, how-

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ever, has been proved rigorously only for $L_p[-\infty, +\infty]$, $p>1$, by E. Hille [6].

Our equations represent the analogue of the Cauchy process on the finite interval. In the terminology of diffusion theory, we may call (0.1) the "backward equation" and (0.2) the "forward equation." Oddly enough, the initial value problems and semi-groups associated with our equations bear a strong resemblance to those connected with the pair of parabolic differential equations

$$u_t = u_{xx} - \frac{x}{1-x^2}u \tag{0.5}$$

and

$$v_t = \frac{\partial}{\partial x}\left\{v_x(x, t) - \frac{x}{1-x^2}v(x, t)\right\}. \tag{0.6}$$

This resemblance accounts for the unexpected similarity between the equations (0.1) and (0.2) and the diffusion equations for the case of regular boundaries. It should be noted that while equations (0.1) and (0.2) are in a sense adjoints to one another, (0.5) and (0.6) are not; in diffusion theory the forward equation to (0.5) would be (0.6) with the first minus sign replaced by a plus sign. We would expect our equations to behave like the heat equation $u_t=u_{xx}$ rather than like (0.5) and (0.6).

To solve the initial value problem for (0.1) and (0.2) we shall follow the pattern used in [3] for the treatment of the parabolic partial differential equations. The integration problem is treated there as an abstract problem for semi-groups. We shall start with (0.1), the backward equation, on the space $C[-1, +1]$. A set $\Sigma\subset D(\Omega)$ is called a *lateral condition* or *conditioning set* if the contraction Ω/Σ is the infinitesimal generator of a contraction semi-group. We show that *all the admissible lateral conditions for the backward equations of diffusion theory have analogues for* (0.1). We shall not prove in this paper that our lateral conditions are the only possible ones, that is, that we have constructed all the contraction semi-groups "generated by Ω" in the terminology of [3]. However, this follows from a generalized version of the proof for the parabolic differential equations which will appear in a forthcoming paper by Feller.

Next, we shall investigate the semi-groups of transformations adjoint to those generated by Ω, cf. §§11, 12, 13. In general, the infinitesimal generator of the adjoint semi-group is given by a contraction of Ω^* plus additional terms. Only with special lateral conditions will Ω^* generate the adjoint semi-group. A similar situation arises in the differential equations' case. The lateral conditions on (0.3) which give a differential operator as the generator of the adjoint contraction semi-group are called the *analogues of the* *classical boundary conditions*. For the singular differential equations (0.5) and (0.6) they are

$$(0.7) \qquad q_j \lim_{x\to(-1)^j} u(t, x) + (-1)^j p_j \lim_{x\to(-1)^j} (1 - x^2)^{1/2} u_x(t, x) = 0$$

and

$$(0.8) \qquad q_j \lim_{x\to(-1)^j} (1 - x^2)^{1/2} v(t, x) + (-1)^j \lim_{x\to(-1)^j} \left(v_x - \frac{x}{1 - x^2} \cdot v\right) = 0$$

respectively, where $p_j \geqq 0$, $q_j \geqq 0$.

In our case, the analogues of (0.7) and (0.8) are

$$(0.9) \qquad q_2 \lim_{x\to 1} u(t, x) + p_2 \lim_{\epsilon\to 0} \pi\epsilon^{-1} \int_{1-\epsilon}^{1} u_x(t, x)(1 - x^2)^{1/2} dx$$

and

$$(0.10) \qquad q_2 \lim_{\epsilon\to 0} \epsilon^{-1/2} \int_{1-\epsilon}^{1} v(t, x) dx + p_2 \pi^{-1} \operatorname{ess.} \lim_{x\to 1} P \int_{-1}^{+1} \frac{v(t, x) dx}{x - t}$$

with similar expressions for the left boundary, cf. Theorem 12.1 where the case $\sigma = \tau = 0$ gives the analogue (0.10).

On the infinite interval, for each $0 < \alpha \leqq 2$, there is a stochastic process, namely the symmetric stable process of order α. The Cauchy process corresponds to $\alpha = 1$, and the diffusion to $\alpha = 2$. A study of these semi-groups for the analogous processes corresponding to $0 < \alpha < 1$ and $1 < \alpha < 2$ on the finite interval will be made in a subsequent paper. Kac [7] has studied the problem for general α, which corresponds to the "absorbing barrier" case of the diffusion equation. Kac's point of departure is the theory of summation of independent random variables(2).

This connection of the stable distributions with diffusion theory has been discovered by Bochner (cf. [2]). Feller [2] has treated this connection from the point of view of Riesz potentials.

The Hille-Yosida theorem [see §1], which gives the necessary and sufficient conditions that an operator on a Banach space X be the infinitesimal generator of a contraction semi-group [cf. Definition 1.3] with range dense in X, leads us to a study of the resolvent equation $\lambda u - \Omega u = h$. We shall show that this equation is equivalent to

$$(0.11) \qquad u(x) + \lambda \int_{-1}^{+1} K(x, y) u(y) dy = \int_{-1}^{+1} K(x, y) h(y) dy,$$

where $K(x, y)$ is defined in (2.6). The homogeneous equation corresponding

(2) In [7] Kac outlines the theory for general α. The detailed theory for $\alpha = 1$ has been developed by Kac and Pollard [8].

to (0.11) appears in [8], but has been used there only for L_2. We shall treat (0.11) in $\mathcal{C}[-1, +1]$ and give proofs for the existence of positive, continuous, norm-decreasing solutions. If an arbitrary, positive, continuous function is put on the right of (0.11), the solutions u are not necessarily positive. However, we can show that if $h \geqq 0$ in (0.11), then so is u. §§1–7 are concerned with the solutions of the resolvent equation.

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1. **The operators Ω and Ω^*.**

DEFINITION 1.1. *Let $g(t)$ be a function such that*

$$\text{essential limit}_{x\to\pm 1}\, P\int_{-1}^{+1} \frac{g(t)dt}{t-x} = l_{\pm 1} \tag{1.1}$$

exists. For such g we define the extended principal value,

$$\overline{P}\cdot\int_{-1}^{+1} \frac{g(t)dt}{x-t}, \tag{1.2}$$

to mean the ordinary Cauchy principal values, cf. [5], *for $|x| < 1$ and $l_{\pm 1}$ for $x = \pm 1$.*

The reason for this definition may be illustrated by taking $g(x) = (1-x^2)^{-1/2}$. The extended principal value is $\equiv 0$ on $[-1, +1]$ but the Cauchy principal value does not exist at ± 1.

DEFINITION 1.2. *The operator Ω from $C[-1, +1]$ to itself is defined by*

$$\Omega f(x) = \pi^{-1}\overline{P}\int_{-1}^{+1} \frac{f'(t)dt}{t-x} \tag{1.3}$$

with domain $D(\Omega)$, the set of all absolutely continuous functions for which the right side is in $C[-1, +1]$.

The operator Ω^* from $L[-1, +1]$ to itself is defined by

$$\Omega^* f(x) = \pi^{-1}\frac{d}{dx}\overline{P}\int_{-1}^{+1} \frac{f(t)dt}{t-x}, \tag{1.4}$$

provided the right side is in $L[-1, +1]$.

LEMMA 1.1. *The domains of Ω and Ω^* are dense in $C[-1, +1]$ and $L[-1, +1]$, respectively.*

Proof. To prove this we use the formula

$$f_n = \frac{1}{\pi}\overline{P}\int_{-1}^{+1} \frac{t^n dt}{(t-x)(1-t^2)^{1/2}} = x^{n-1} + \alpha_2 x^{n-2} + \cdots + \alpha_n \tag{1.5}$$

where

$$\alpha_n = 0, \qquad n \text{ even},$$

$$(1.6) \qquad \alpha_n = \frac{1}{\pi}\int_{-1}^{+1} \frac{t^{n-1}dt}{(1-t^2)^{1/2}} = \frac{1\cdot 3\cdot 5 \cdots n-2}{2\cdot 4\cdot 6\cdots n-1}, \qquad n \text{ odd},$$

$$\alpha_1 = 1.$$

This formula is proved using the fact that $f_0=0$, cf. [4, last chapter], and the simple recursion relation

$$(1.7) \qquad f_n = xf_{n-1} + \alpha_n.$$

From (1.5) we see that $D(\Omega)$ contains all functions of the form $f(x) = p_n(x)(1-x^2)^{1/2}+A\cdot\text{arc sin } x+B$, where A and B are constants and $p_n(x)$ is a polynomial of degree n. This set is clearly dense in $C[-1, +1]$. Similarly, $D(\Omega^*)$ contains all functions of the form $p_n(x)(1-x^2)^{-1/2}$.

If $g\in L[-1, +1]$, then $P\cdot\int_{-1}^{+1}g(t)dt/(t-x)$ exists for almost all $x \in[-1, +1]$, cf. [10, chap. V]. Another useful relation which we shall employ is contained in the following lemma.

LEMMA 1.2. *The transformation*

$$(1.8) \qquad Tf(t) = \frac{1}{\pi} P\int_{-1}^{+1} \frac{f(x)(1-x^2)^{1/2}dx}{x-t}$$

is a continuous transformation from the space $H[(1-x^2)^{1/2}]$ of f such that $\int_{-1}^{+1}f^2(1-x^2)^{1/2}dx<\infty$ to the space $H[(1-x^2)^{-1/2}]$. Furthermore, $T\cdot P_n=Q_{n+1}$ where $\{P_n\}$ and $\{Q_n\}$ denote the Jacobi polynomials which are orthonormal with respect to $(1-x^2)^{1/2}$ and $(1-x^2)^{-1/2}$, respectively.

Proof. That $TP_n=Q_{n+1}$ is proved by Ahiezer in [1]. The continuity follows easily from the fact that if $f\in H[(1-x^2)^{1/2}]$, then $f(1-x^2)^{1/2} \in L_2[-1, +1]$. The transformation is continuous in L_2 (cf. [10, chap. V]. If we put $f=\sum_{k=1}^{\infty} a_kP_k(x)$ it is clear that $T(\sum_{k=1}^{n} a_kP_k(x))$ converges in the norm of H, and from the L_2 continuity it follows that it must converge to Tf.

DEFINITION 1.3. *Let X be a Banach space in which positive elements are defined. A semi-group of linear transformations from X to itself is called a contraction semi-group if*

$$(1.9) \qquad \|T_t\| \leqq \|x\|,$$

$$(1.10) \qquad \|T_{t+h}x - T_tx\| \to 0 \qquad \textit{as } h\downarrow 0,\ t\geqq 0,$$

and if

$$(1.11) \qquad x\geqq 0 \quad \textit{implies} \quad T_tx\geqq 0.$$

THEOREM 1.1. (THE HILLE-YOSIDA THEOREM.) *The infinitesimal generator*

A of a contraction semi-group on X with range dense in X is a linear operator with domain dense in X. Furthermore, to each $x \in X$ *and each* $\lambda > 0$, *there exists a unique* $y_\lambda \in D(A)$ *such that*

$$\lambda \|y_\lambda\| \leqq \|x\|, \tag{1.12}$$

$$\lambda y_\lambda - A y_\lambda = x, \tag{1.13}$$

and

$$y_\lambda \geqq 0 \quad \textit{when} \quad x \geqq 0. \tag{1.14}$$

Conversely, a linear operator with these properties is the infinitesimal generator of a contraction semi-group with range dense in X.

In our case Ω itself is not an infinitesimal generator since $\lambda u - \Omega u = 0$ has solutions, cf. §7. However, Ω may be the *extension* of an infinitesimal generator. The problem of finding the most general lateral condition is to determine all sets Σ (independent of λ) such that $\lambda y_\lambda - \Omega y_\lambda = x$ has a unique solution for $x \in C[-1, +1]$, satisfying (1.1) and (1.14). Then the contraction Ω/Σ is the infinitesimal generator of a contraction semi-group.

In case the closure $\bar{\Sigma}$ is a proper subspace of $C[-1, +1]$, we must take $\bar{\Sigma} = X$ in the Hille-Yosida Theorem in order that the conclusion of that theorem hold. The restriction that the range of the semi-group be dense in X is equivalent to the statement that $\|T_t x - x\| \to 0$ as $t \to 0$ for all $x \in X$, cf. [3]. Hence all our solutions converge in norm to the initial values. This does not mean that the semi-group admits no extension from $\bar{\Sigma}$ to the whole space but the extensions will converge only pointwise to the initial value.

It might be pointed out that a certain *duality* exists between the solutions of $\lambda u - \Omega u = 0$ in $C[-1, +1]$ and those of $\lambda v - \Omega^* v = 0$ in $L[-1, +1]$. If v is a solution of the latter, then there exists an a such that $u = \int_a^x v(t) \cdot dt$ is a solution of the former, and conversely, if u is a solution of $\lambda u - \Omega u = 0$, then u is absolutely continuous (cf. Theorem 2.1) and u' is a solution of $\lambda v - \Omega^* v = 0$.

2. **Formal reduction to an integral equation.** In order to apply the Hille-Yosida theorem we must make a study of the equation

$$\lambda f(x) - \frac{1}{\pi} \overline{P} \int_{-1}^{+1} \frac{f'(t)}{t - x} dt = h(x) \tag{2.1}$$

where $h(x)$ is given in $C[-1, +1]$. In this section we reduce (2.1) to an integral equation in a purely formal way. The proof will be carried out in §3.

The following lemma, due to H. Söhngen [9], makes this reduction possible.

LEMMA 2.1. *If* $g(x)$ *and* $g^2(x)(1-x^2)^{1/2}$ *are integrable in* $[-1, +1]$, *then the only solutions of*

$$g(x) = \frac{1}{\pi} P \int_{-1}^{+1} \frac{f(t)dt}{t - x}, \qquad -1 < x < +1, \tag{2.2}$$

such that $f(x)$ and $f^2(x)(1-x^2)^{1/2}$ are integrable in $[-1, +1]$ are given by

$$f(x) = \frac{A}{(1 - x^2)^{1/2}} - \frac{1}{\pi(1 - x^2)^{1/2}} P \int_{-1}^{+1} \frac{g(t)(1 - t^2)^{1/2}}{t - x} dt \qquad a.e. \tag{2.3}$$

where A is an arbitrary constant.

Applying this inversion formula to (2.1) we obtain

$$\begin{aligned} f'(x) &= \frac{A}{(1 - x^2)^{1/2}} \\ &\quad - \frac{1}{\pi(1 - x^2)^{1/2}} P \int_{-1}^{+1} \frac{\{\lambda f(y) - h(y)\} \cdot (1 - y^2)^{1/2}}{y - x} dy \qquad \text{a.e.} \end{aligned} \tag{2.4}$$

Then, integrating (2.4) formally, we get

$$f(x) + \lambda \int_{-1}^{+1} K(x, y) f(y) dy = \int_{-1}^{+1} K(x, y) h(y) dy + A \text{ arc sin } x + B \tag{2.5}$$

where

$$K(x, y) = \frac{1}{2\pi} \log \left\{ \frac{1 - xy + ((1 - x^2)(1 - y^2))^{1/2}}{1 - xy - ((1 - x^2)(1 - y^2))^{1/2}} \right\}. \tag{2.6}$$

It will be seen in §§4 and 6 that we can write the solution to (2.5) in terms of a solving kernel $\Gamma(x, y; \lambda)$. This gives

$$f(x) = H(x) - \lambda \int_{-1}^{+1} \Gamma(x, y; \lambda) \cdot H(y) dy + A\phi_1(x) + B\phi_2(x) \tag{2.7}$$

where

$$\begin{aligned} H(x) &= \int_{-1}^{+1} K(x, y) h(y) dy, \\ \phi_1(x) &= \text{arc sin } x - \lambda \int_{-1}^{+1} \Gamma(x, y; \lambda) \text{ arc sin } y dy, \\ \phi_2(x) &= \frac{1}{2} - \frac{\lambda}{2} \int_{-1}^{+1} \Gamma(x, y; \lambda) dy \end{aligned} \tag{2.8}$$

with arc sin $\pm 1 = \pm \pi/2$. It will be shown in §7 that $\phi_1(x)$ and $\phi_2(x)$ are solutions of the homogeneous equation, i.e. the equation (2.1) with $h(x) \equiv 0$.

In view of the fact that the resolvent kernel satisfies

$$\Gamma(x, y) + \lambda \int_{-1}^{+1} K(x, \rho)\Gamma(\rho, y)d\rho = K(x, y), \tag{2.9}$$

we may rewrite (2.7) as

$$f(x) = \int_{-1}^{+1} \Gamma(x, y)h(y)dy + A\phi_1(x) + B\phi_2(x). \tag{2.10}$$

We have omitted the λ in $\Gamma(x, y; \lambda)$ for simplicity.

The precise statement of this equivalence is contained in the next theorem.

THEOREM 2.1. *If $h \in C[-1, +1]$, then every absolutely continuous solution of* (2.1) *is also a solution of* (2.5) *for some A and B. Conversely, every continuous solution of* (2.5) *is an absolutely continuous solution of* (2.1).

The proof of Theorem 2.1 is given in the next section.

3. Proof of Theorem 2.1.

LEMMA 3.1. *If $f_n(x)$ is a sequence of measurable functions such that $f_n(x) \to f(x)$ as $n \to \infty$ for almost all $x \in [-1, +1]$, and*

$$\int_{-1}^{+1} | f_n(x) |^p dx < M \tag{3.1}$$

for some $p > 1$, and if $g(x) \in L_{p'}[-1, +1]$ where $1/p + 1/p' = 1$, then

$$\lim_{n\to\infty} \int_{-1}^{+1} f_n(x)g(x)dx = \int_{-1}^{+1} f(x)g(x)dx. \tag{3.2}$$

Proof. The proof is given in [10, chap. V].

LEMMA 3.2. *If $g \in C[-1, +1]$, then*

$$\int_{-1}^{x} \frac{1}{(1 - t^2)^{1/2}} \left\{ P \frac{1}{\pi} \int_{-1}^{+1} \frac{g(y)(1 - y^2)^{1/2}dy}{y - t} \right\} dt = \int_{-1}^{+1} K(x, y)g(y)dy, \tag{3.3}$$

where $K(x, y)$ is the function defined in (2.6).

Proof. Since we do not assume that $g(x)$ is differentiable, the condition given by Hardy in [5b] for the interchange of the order of integration does not apply. However, formula (3.3) may be verified as follows. We write

$$\begin{aligned} &\frac{1}{\pi} P \int_{-1}^{+1} \frac{g(y)(1 - y^2)^{1/2}dy}{y - x} \\ &\qquad = \lim_{\xi\to 0} \frac{1}{\pi} \int_{-1}^{+1} \frac{g(y)(1 - y^2)^{1/2}(y - x)dy}{(y - x)^2 + \xi^2} \qquad \text{a.e.} \end{aligned} \tag{3.4}$$

Equation (3.4) holds almost everywhere on $[-1, +1]$, cf. [10, chap. V].

It is well known from the theory of conjugate functions in L_p, cf. [10, chap. V], that for any $p>1$,

$$\int_{-1}^{x}\left|\frac{1}{\pi}\int_{-1}^{+1}\frac{g(y)(1-y^2)^{1/2}(y-t)dy}{(y-t)^2+\xi^2}\right|^p dt \tag{3.5}$$
$$\leqq M_p\int_{-1}^{x}\left|g(y)(1-y^2)^{1/2}\right|^p dy$$

where M_p is a constant depending only on p. We choose $p>2$. Then since $(1-x^2)^{-1/2}\in L_{p'}[-1,+1]$, we may apply Lemma 3.1 to conclude that

$$\begin{aligned}
&\int_{-1}^{x}\frac{1}{(1-t^2)^{1/2}}\left\{\overline{P}\,\frac{1}{\pi}\int_{-1}^{+1}\frac{g(y)(1-y^2)^{1/2}dy}{y-t}\right\}dt\\
&=\lim_{\xi\to 0}\int_{-1}^{x}\frac{1}{(1-t^2)^{1/2}}\left\{\frac{1}{\pi}\int_{-1}^{+1}\frac{g(y)(1-y^2)^{1/2}(y-t)dy}{(y-t)^2+\xi^2}\right\}dt\\
&=\lim_{\xi\to 0}\int_{-1}^{+1}g(y)(1-y^2)^{1/2}\left\{\frac{1}{\pi}\int_{-1}^{x}\frac{1}{(1-t^2)^{1/2}}\frac{y-t}{(y-t)^2+\xi^2}dt\right\}dy \qquad (3.6)\\
&=\int_{-1}^{+1}g(y)(1-y^2)^{1/2}\left\{\overline{P}\,\frac{1}{\pi}\int_{-1}^{x}\frac{1}{(1-t^2)^{1/2}}\frac{dt}{y-t}\right\}dy\\
&=\int_{-1}^{+1}g(y)K(x,y)dy.
\end{aligned}$$

Proof of Theorem 2.1. Suppose first that $f(x)$ is an absolutely continuous solution of (2.1). Lemma 2.1 leads us to (2.4), and (2.5) follows from Lemma 3.2 and the assumption of absolute continuity; the steps are reversible.

4. **The integral equation.** The kernel $K(x, y)$ is symmetric, non-negative, and $K(\pm 1, y)=K(x, \pm 1)=0$ for $x, y\in(-1, 1)$. In this section we state theorems about the equation

$$f(x)+\lambda\int_{-1}^{+1}f(y)K(x,y)dy=\int_{-1}^{+1}h(y)K(x,y)dy. \tag{4.1}$$

They will be proved in the next section. The step from (4.1) to the more general (2.5) will prove easy, cf. §8. In the terminology which is usual in diffusion theory, we say that the solution $f(x)$ of (4.1) is the *absorbing barrier* solution of (2.1), that is, $f(x)$ is the solution for which $\lim_{x\to\pm 1} f(x)=0$.

THEOREM 4.1. *For each* $h(x)\in C[-1, +1]$ *and each* $\lambda>0$, *the equation* (4.1) *has a unique solution* $f(x)\in C[-1, +1]$ *and* $f(1)=f(-1)=0$. *If* $h(x)\geqq 0$, *then* $f(x)\geqq 0$.

THEOREM 4.2. *The solution described in Theorem* 4.1 *can be written in the form*

$$f(x) = \int_{-1}^{+1} \Gamma(x, y; \lambda) h(y) dy \tag{4.2}$$

where $\Gamma(x, y; \lambda)$ *has the following properties*:

(4.3) (a) $\Gamma(x, y; \lambda)$ *is symmetric and continuous in* y *for fixed* x *except when* $x = y$.
(b) $\Gamma(x, y; \lambda) \geqq 0$. (c) *Formula* (2.9) *holds.*

Furthermore

$$\lambda \|f\| \leqq \|h\|. \tag{4.4}$$

DEFINITION 4.1. *We define*

$$\begin{aligned} \delta_2 f &= \frac{\pi}{2^{1/2}} \left\{ \lim_{\epsilon \to 0} \frac{f(1) - f(1-\epsilon)}{\epsilon^{1/2}} \right\}, \\ \delta_1 f &= \frac{\pi}{2^{1/2}} \left\{ \lim_{\epsilon \to 0} \frac{f(-1+\epsilon) - f(-1)}{\epsilon^{1/2}} \right\}. \end{aligned} \tag{4.5}$$

THEOREM 4.3. *If* $h \in C[-1, +1]$ *and* $f(x)$ *is the solution described in Theorem* 4.1, *then* $\delta_j f$ *exists and is given by*

$$\delta_j f = (-1)^{j+1} \int_{-1}^{+1} h(x) \xi_j^*(x) dx, \qquad j = 1, 2, \tag{4.6}$$

where

$$\xi_j^*(x) = \left(\frac{1-x}{1+x} \right)^{(-1)^{j+1}/2} - \lambda \int_{-1}^{+1} \Gamma(x, y) \left(\frac{1-y}{1+y} \right)^{(-1)^{j+1}/2} dy. \tag{4.7}$$

5. **Preparatory lemmas.** The lemmas of this section will be used to prove the theorems of §4.

LEMMA 5.1. *If* $K(x, y)$ *is defined as in* (2.6), *then*

$$\begin{aligned} \int_a^b K(x, y) dy &= (b - x) K(x, b) + (x - a) K(x, a) \\ &\quad + \pi^{-1}(1 - x^2)^{1/2}(\arcsin b - \arcsin a). \end{aligned} \tag{5.1}$$

Proof. Integrate by parts.

The following lemma will not be used until later sections, but for convenience we prove it here.

LEMMA 5.2. *For each* $\epsilon > 0$ *we can choose an* $\eta > 0$, *independent of* x, *so that*

$$\int_{R_\eta} K(x, y) dy < \epsilon \tag{5.2}$$

for $|x| \leqq 1$, *where* R_η *is the region defined by* $|x-y| < \eta$ *and* $|y| \leqq 1$.

Proof. From Lemma 5.1 we have for $-1+\eta < x < 1-\eta$,

$$\int_{R_\eta} K(x, y)dy = \eta K(x, x+\eta) + \eta K(x, x-\eta) + \pi^{-1}(1-x^2)^{1/2}[\arcsin(x+\eta) - \arcsin(x-\eta)], \tag{5.3}$$

while for $1-\eta < x < 1$,

$$\int_{R_\eta} K(x, y)dy = \eta K(x, x-\eta) + \pi^{-1}(1-x^2)^{1/2}[\arcsin(x+\eta) - \arcsin(x-\eta)]. \tag{5.4}$$

A similar formula holds for $-1 < x < -1+\eta$. Clearly, it is sufficient to show that given $\delta > 0$, we can find $\eta > 0$ so that $\eta K(x, x\pm\eta) < \delta$ for all $x \in [-1, +1]$.

We write the kernel in the form,

$$K(x, y) = \frac{1}{2\pi} \log \left\{ \frac{((1-x^2)^{1/2} + (1-y^2)^{1/2})^2 + (x-y)^2}{((1-x^2)^{1/2} - (1-y^2)^{1/2})^2 + (x-y)^2} \right\}. \tag{5.5}$$

Since $\eta^2 < ((1-x^2)^{1/2} \pm (1-y^2)^{1/2})^2 + \eta^2 < 4+\eta^2$ for x and y in $[-1, +1]$, the lemma is proved.

LEMMA 5.3. *If* $f(x)$ *is bounded on* $[-1, +1]$ *and*

$$\begin{aligned} u(x) &= 0, & |x| &> 1, \\ u(x) &= \int_{-1}^{+1} K(x, y)f(y)dy, & -1 &\leqq x \leqq 1, \end{aligned} \tag{5.6}$$

then there exists a $\delta > 0$, *independent of* x, *such that for* $|h| < \delta$, *and all* $0 < \alpha < 1/2$,

$$|u(x+h) - u(x)| < M_\alpha |h|^\alpha, \qquad -\infty < x < \infty, \tag{5.7}$$

where M_α *is a constant depending only on* α.

Proof. Assume $h > 0$. Then for $|x+h| < 1$,

$$\begin{aligned} \|f\|^{-1} |u(x+h) - u(x)| &\leqq \int_{-1}^{+1} |K(x+h, y) - K(x, y)| dy \\ &\leqq \int_{-1}^{x} K(x, y) - K(x+h, y)dy + \int_{x}^{x+h} K(x, y) + K(x+h, y)dy \\ &\quad + \int_{x+h}^{1} K(x+h, y) - K(x, y)dy. \end{aligned} \tag{5.8}$$

The first and second integrals are given by

$$(5.9)\qquad hK(x+h, x) + \frac{1}{\pi}\left(\arcsin x - \frac{\pi}{2}\right)[R(x) - R(x+h)]$$

and

$$(5.10)\qquad 2hK(x, x+h) + \frac{1}{\pi}[\arcsin (x+h) - \arcsin x][R(x) + R(x+h)]$$

respectively, where $R(x)=(1-x^2)^{1/2}$. The third integral is similar to the first. From (5.5) we see that $h^{1-\alpha}K(x, x+h)<\epsilon$ for h sufficiently small. Furthermore,

$$(5.11)\ \arcsin (x+h) - \arcsin x = \int_x^{x+h} (1-t^2)^{-1/2}dt \leqq \|(1-x^2)^{-1/2}\|_p h^{1-1/p}$$

for all $1<p<2$. A similar estimate holds for $R(x)-R(x+h)$. For $|x+h|>1$, $|x|\leqq 1$, we have $|u(x+h)-u(x)| = |u(x)| \leqq \|f\|(1-x^2)^{1/2}$. This completes the proof of the lemma.

LEMMA 5.4. *If $g(x)$ satisfies* (5.7) *for some* $0<\alpha<1/2$ *on* $[-1, 1]$, *then*

$$(5.12)\qquad v(x) = P\int_{-1}^{+1} \frac{g(y)(1-y^2)^{1/2}}{y-x}dy$$

satisfies (5.7) *for all* $x\in[-\infty, +\infty]$.

Proof. The function defined by $g(x)(1-x^2)^{1/2}$ on $[-1, +1]$ and zero elsewhere satisfies (5.7) for all $x\in[-\infty, +\infty]$. The lemma then follows from a well known result proved in [10, p. 145].

LEMMA 5.5. *If $f(x)$ satisfies* (5.7) *for some* $0<\alpha<1/2$, *then $u(x)$ defined in* (5.6) *has a continuous derivative in* $(-1, +1)$.

Proof. We have by (3.3)

$$(5.13)\qquad u(x) = \int_{-1}^{x} \frac{dt}{(1-t^2)^{1/2}} P\int_{-1}^{+1} \frac{f(y)(1-y^2)^{1/2}}{y-t}dy,$$

so that $u'(x)$ is equal to the integrand, which is continuous in $(-1, +1)$ by Lemma 5.4.

LEMMA 5.6. *If $h(x)$ satisfies* (5.7) *and* $f(x)\in C[-1, +1]$ *satisfies* (4.1), *then*

$$(5.14)\qquad F(x) = P\int_{-1}^{+1} \frac{f(t)}{t-x}dt$$

is absolutely continuous in $[-\infty, +\infty]$.

Proof. We first note that $f(x)$ satisfies (5.7) by Lemma 5.3. Furthermore by Theorem 2.1, equation (2.4), and Lemma 5.4, it follows that $f(x)$ has a continuous derivative in $(-1, +1)$. We next show that

$$F(x) = -\int_{-1}^{+1} f'(t) \log |t - x| \, dt, \qquad -1 < x < 1. \tag{5.15}$$

First,

$$\int_{x+\epsilon}^{1} + \int_{-1}^{x-\epsilon} \frac{f(t)}{t-x} dt = -\int_{x+\epsilon}^{1} + \int_{-1}^{x-\epsilon} f'(t) \log |t - x| \, dt \\ + [f(x+\epsilon) - f(x-\epsilon)] \log \epsilon. \tag{5.16}$$

Since f satisfies (5.7), we obtain (5.15) upon letting $\epsilon \to 0$. Moreover, we can show that

$$F'(x) = P\int_{-1}^{+1} \frac{f'(t)}{t-x} \cdot dt, \qquad -1 < x < 1, \tag{5.17}$$

which is continuous in $(-1, +1)$ since f satisfies (2.1). To prove (5.17), note that for $-1<x<1$,

$$\begin{aligned} F(x+h) - F(x) &= -\int_{-1}^{+1} f'(t) dt P \int_{x}^{x+h} \frac{du}{u-t} \\ &= -\int_{-1}^{+1} f'(t) \left[\lim_{\epsilon \to 0} \int_{x}^{x+h} \frac{u-t}{(u-t)^2 + \epsilon^2} \right] dt. \end{aligned} \tag{5.18}$$

Since $f'(t)(1-t^2)^{1/2}$ is continuous, it follows that $f'(t) \in L_p[-1, +1]$ for all $1<p<2$. Hence by Lemma 3.1 and the argument used in the proof of Lemma 3.2,

$$\begin{aligned} F(x+h) - F(x) &= \lim_{\epsilon \to 0} -\int_{-1}^{+1} f'(t) dt \int_{x}^{x+h} \frac{u-t}{(u-t)^2+\epsilon^2} du \\ &= \lim_{\epsilon \to 0} -\int_{x}^{x+h} du \int_{-1}^{+1} \frac{f'(t)(u-t)}{(u-t)^2+\epsilon^2} dt \\ &= \int_{x}^{x+h} du P \int_{-1}^{+1} \frac{f'(t)}{t-u} dt. \end{aligned} \tag{5.19}$$

Since the integrand is continuous at x, we have (5.17).

We have now shown that $F(x)$ has a continuous derivative in $(-1, +1)$. It is clear that $F'(x)$ is continuous for $|x| > 1$. Furthermore $\lim_{x \to 1^-} F'(x) = \lim_{x \to 1^-} h(x) < \infty$ by (2.1) and Theorem 2.1 and $\lim_{x \to 1^+} (x^2-1)^{1-\alpha} F'(x) < \infty$ by (5.17). A similar statement holds at -1. From Lemma 5.3 the function defined by $f(x)$ on $[-1, +1]$ and 0 elsewhere satisfies (5.7) on $[-\infty, +\infty]$.

Hence from the result quoted before in [10, p. 145], it follows that $F(x)$ satisfies (5.7) on $[-\infty, +\infty]$, and, in particular, at ± 1. Combining the results of this paragraph, we conclude that $F(x)$ is absolutely continuous in $[-\infty, +\infty]$.

LEMMA 5.7. *If $f(x) \geqq 0$ ($f \not\equiv 0$) satisfies (5.7) for $x \in [-1, +1]$ and $f(x) \equiv 0$ in $[a, b]$ where $-1 < a < b < 1$, then $u(x)$ defined by (5.6) has no relative maximum in $[a, b]$.*

Proof. For $x \in (a, b)$,

$$u''(x)(1 - x^2) - u'(x)\cdot x \cdot (1 - x^2)^{-1/2} = \int_{-1}^{a} + \int_{b}^{1} \frac{f(y)(1 - y^2)^{1/2}}{(y - x)^2}\, dy > 0. \tag{5.20}$$

If $u'(x) = 0$ for some $x \in (a, b)$, then (5.20) implies that $u''(x) > 0$. Hence there is no maximum in (a, b). Now suppose that $u'(b) = 0$. Then

$$\begin{aligned} u''(b - \epsilon)[1 - (b - \epsilon)^2]^{1/2} &> \int_{-1}^{a} \frac{f(y)(1 - y^2)^{1/2}}{(y - b)^2}\, dy + \int_{b}^{1} \frac{f(y)(1 - y^2)^{1/2}}{(y - a)^2}\, dy \\ &\quad + u'(b - \epsilon)\, \frac{(b - \epsilon)}{[1 - (b - \epsilon)^2]^{1/2}} \end{aligned} \tag{5.21}$$

which is >0 for ϵ sufficiently small since $\lim_{\epsilon \to 0} u'(b - \epsilon) = 0$ by Lemma 5.5. Therefore, there is no maximum at $x = b$. A similar argument excludes the possibility of a maximum at $x = a$.

LEMMA 5.8. *If $\phi(x) \in C[-1, +1]$ and*

$$\phi(x) + \lambda \int_{-1}^{+1} K(x, y)\phi(y)dy = A \tag{5.22}$$

for some $\lambda \geqq 0$, $A \geqq 0$, then $\phi(x) \geqq 0$ on $[-1, +1]$.

Proof. From Lemmas 5.3 and 5.5, it follows that $\phi(x)$ has a continuous derivative in $(-1, +1)$. We split ϕ into its positive and negative parts, i.e. set $\phi = \phi^+ - \phi^-$ and assume that $\phi^- \not\equiv 0$. Put

$$\begin{aligned} P(x) &= \int_{-1}^{+1} K(x, y)\phi^+(y)dy, \\ N(x) &= \int_{-1}^{+1} K(x, y)\phi^-(y)dy. \end{aligned} \tag{5.23}$$

Since ϕ^- also satisfies (5.7), it follows that $N(x)$ has a continuous derivative in $(-1, +1)$. The maximum of $N(x)$ is assumed at a point c in $(-1, +1)$. By Lemma 5.7, the point c cannot belong to an interval in which $\phi^- \equiv 0$. Hence, there is an interval (a, b) such that

$$(5.24)\qquad \begin{aligned} &-1 < a < c < b < 1, && \\ &\phi^-(x) > 0 && \text{for } x \text{ in } (a, b), \\ &\phi^-(a) = \phi^-(b) = 0, && \\ &\phi^+(x) = 0 && \text{for } x \text{ in } [a, b]. \end{aligned}$$

We also have

$$(5.25)\qquad \begin{aligned} A &= \phi(c) + \lambda \int_{-1}^{+1} K(c, y)\phi(y)dy \\ &= -\phi^-(c) + \lambda P(c) - \lambda N(c) < \lambda P(c) - \lambda N(c). \end{aligned}$$

By Lemma 5.7, the function $P(x)$ can have no maximum in (a, b). Hence, the largest value of $P(x)$ in $[a, b]$ is assumed at $x = a$ or $x = b$, say $x = b$. Then

$$(5.26)\qquad \begin{aligned} \lambda P(c) - \lambda N(c) &\leqq \lambda P(b) - \lambda N(c) \leqq \lambda P(b) - \lambda N(b) \\ &= \phi(b) + \lambda \int_{-1}^{+1} K(b, y)\phi(y)dy = A, \end{aligned}$$

which is impossible in view of (5.25).

6. **Proof of Theorems 4.1, 4.2, and 4.3.**

(a) **Proof of Theorem 4.1.** By Lemma 5.8, there can be no negative eigenvalues associated with the kernel $K(x, y)$. The first iterated kernel $K^{(2)}(x, y)$ is continuous, and if $\phi_\nu(x)$ is the eigenfunction corresponding to $-\lambda_\nu$, we can write the solution to (4.1) as

$$(6.1)\qquad f(x) = \int_{-1}^{+1} K(x, y)h(y)dy - \lambda \sum_{\nu=1}^{\infty} \frac{h_\nu \phi_\nu(x)}{(\lambda_\nu - \lambda)\lambda_\nu}$$

with $h_\nu = \int_{-1}^{+1} h(x)\phi_\nu(x)dx$. The series in (6.1) converges uniformly. By Lemma 5.3, if $h \in C[-1, +1]$ then $f \in C[-1, +1]$. Furthermore, $|f(x)| \leqq \|\lambda h - f\| \cdot (1 - x^2)^{1/2}$, so that $\lim_{x \to \pm 1} f(x) = 0$.

We must prove that $h \geqq 0$ implies $f \geqq 0$[3]. For each $\tau > 0$, we define an operator A_τ on the space $L_p[-1, +1]$ for some $1 < p < 2$ by

$$(6.2)\qquad A_\tau f = \int_{-1}^{+1} \frac{f(t)}{(t - x)^2 + \tau^2} dt - \frac{f(x)}{\tau} = H_\tau f - \frac{f}{\tau} \cdot$$

The operator A_τ is bounded on $L_p[-1, +1]$ and is the infinitesimal generator of the semi-group

(3) The proof for the positivity of $(\lambda I - \Omega)^{-1}$ along these lines was suggested by W. Feller.

(6.3) $$T_t f = e^{-t/\tau} \sum_{n=0}^{\infty} \frac{H_\tau^n}{n!} f.$$

Let $h \in C[-1, +1]$ satisfy (5.7) and define $f \in C[-1, +1]$ and f_τ by

(6.4) $$\lambda f_\tau - A_\tau f_\tau = h, \qquad \lambda f - \Omega f = h$$

where Ω is defined in (1.3). Next note that

(6.5) $$\lambda(f_\tau - f) - A_\tau(f_\tau - f) = (A_\tau - \Omega) f.$$

From 3.3 and Lemma 5.4, we see that $f'(x)(1-x^2)^{1/2} \in C[-1, +1]$ and hence $f' \in L_p[-1, +1]$. By definition it follows that $\Omega f \in C[-1, +1]$. These two conditions plus Lemma 5.6 imply $\text{l.i.m.}_{\tau \to 0} (A_\tau - \Omega) f = 0$ by a theorem of Hille [6]. We may write

(6.6) $$(f_\tau - f) = (\lambda I - A_\tau)^{-1}(A_\tau - \Omega) f.$$

By the Hille-Yosida Theorem

(6.7) $$\| f_\tau - f \|_p \leqq \lambda^{-1} \| (A_\tau - \Omega) f \|_p,$$

which proves that $f_\tau \to f$ in $L_p[-1, +1]$. Since the h satisfying (5.7) are dense in $L_p[-1, +1]$ and $|(\lambda I - A_\tau)^{-1}| < \lambda^{-1}$, it follows that for all $h \in C[-1, +1]$,

(6.8) $$\underset{\tau \to 0}{\text{l.i.m.}} (\lambda I - A_\tau)^{-1} h = (\lambda I - \Omega)^{-1} h.$$

Since H_τ is a positivity preserving transformation, so is $(\lambda I - A_\tau)^{-1}$. From (6.8), we conclude $(\lambda I - \Omega)^{-1}$ is positivity preserving. This completes the proof of Theorem 4.1.

(b) **Proof of Theorem 4.2.** The solving kernel corresponding to $K(x, y)$ is given by

(6.9) $$\Gamma(x, y; \lambda) = K(x, y) - \lambda \sum_{\nu=1}^{\infty} \frac{\phi_\nu(x)\phi_\nu(y)}{(\lambda_\nu - \lambda)\lambda_\nu}$$

where $\phi_\nu(x)$ is the eigenfunction corresponding to $-\lambda_\nu$. The only part of this theorem which is not immediate is (4.4). To prove this consider $\phi(x)$ defined by (5.22) with $A = 1$. We can write ϕ as

(6.10) $$\phi(x) = 1 - \lambda \int_{-1}^{+1} \Gamma(x, y; \lambda) dy.$$

From Lemma 5.8, we have $0 \leqq \phi(x) \leqq 1$. Then since $\Gamma(x, y; \lambda) \geqq 0$ from Theorem 4.1,

(6.11) $$\lambda | f(x) | \leqq \lambda \| h \| \int_{-1}^{+1} \Gamma(x, y; \lambda) dy \leqq \| h \|.$$

These two theorems show that Ω operating on the Banach space X of func-

tions which are continuous on $[-1, +1]$ *and which vanish at* ± 1 *is the infinitesimal generator of a contraction semi-group, since the conditions of Theorem* 1.1 *are satisfied.*

(c) **Proof of Theorem 4.3.** From (4.1)

$$\frac{f(1-\epsilon)}{\epsilon^{1/2}} = \int_{-1}^{+1} \{h(y) - \lambda f(y)\} \frac{K(1-\epsilon, y)dy}{\epsilon^{1/2}}. \tag{6.12}$$

Using Lemma 5.2 and (2.6) it is easily shown that

$$\lim_{\epsilon\to 0} \frac{f(1-\epsilon)}{\epsilon^{1/2}} = -2^{1/2} \int_{-1}^{+1} \{h(y) - \lambda f(y)\} \left(\frac{1+y}{1-y}\right)^{1/2} dy. \tag{6.13}$$

Upon substituting (4.2) for f and changing the order of integration, we get (4.5). The result for $\delta_2 f$ is proved the same way.

7. **The equation** $\lambda F - \Omega F = 0$. Before we study the resolvent in the general case we must have some information about the solutions of

$$\lambda f(x) - \frac{1}{\pi} \overline{P} \int_{-1}^{+1} \frac{f'(t)dt}{t-x} = 0. \tag{7.1}$$

THEOREM 7.1. *For each* $\lambda > 0$, *equation* (7.1) *has two absolutely continuous independent solutions*:

$$\begin{aligned} \xi_1(x) &= \lambda^{-1}[\phi_2(x) - \pi^{-1}\phi_1(x)], \\ \xi_2(x) &= \lambda^{-1}[\phi_2(x) + \pi^{-1}\phi_1(x)] \end{aligned} \tag{7.2}$$

where ϕ_1 *and* ϕ_2 *are defined in* (2.8). *Every other solution in* $C[-1, +1]$ *is a linear combination of* ξ_1 *and* ξ_2.

Proof. That ϕ_1 and ϕ_2 are solutions of (7.1) follows from Theorems 4.1 and 4.2 and the fact that from (1.5)

$$\overline{P} \int_{-1}^{+1} \frac{1}{(1-t^2)^{1/2}(t-x)} dt = 0. \tag{7.3}$$

Thus ξ_1 and ξ_2 are obviously solutions. Suppose ξ_3 is a third solution. Then the function f defined by

$$f(x) = \lambda \cdot \xi_3(+1) \cdot \xi_2(x) + \lambda \xi_3(-1) \cdot \xi_1(x) - \xi_3(x) \tag{7.4}$$

is a solution of (7.1) which vanishes at $x = \pm 1$. Hence by Theorems 2.1 and 4.2, $f(x) \equiv 0$.

THEOREM 7.2. *The functions* ξ_1 *and* ξ_2 *defined by* (7.2) *are non-negative and*

$$\|\xi_i\| \leqq \lambda^{-1}, \qquad i = 1, 2. \tag{7.5}$$

Proof. Let

$$\eta_1(x) = -\pi^{-1} \arc \sin x + 1/2, \qquad \eta_2(x) = \pi^{-1} \arc \sin x + 1/2 \tag{7.6}$$

where arc sin $\pm 1 = \pm\pi/2$. Then

$$\xi_i(x) = -\int_{-1}^{+1} \Gamma(x, y)\eta_i(y)dy + \lambda^{-1}\eta_i(x). \tag{7.7}$$

Since $\eta_i \geqq 0$, it follows from Theorem 4.1 that $\xi_i \geqq 0$. Upon adding the two equations in (7.7) and noting that $\eta_1 + \eta_2 \equiv 1$ we obtain

$$\lambda^{-1} - \xi_1(x) - \xi_2(x) = \int_{-1}^{+1} \Gamma(x, y)dy \geqq 0, \tag{7.8}$$

which proves (7.5).

8. **The resolvent for one absorbing boundary.** The general solution in $C[-1, +1]$ of (2.1) for $\lambda > 0$ can be written as

$$f(x) = \int_{-1}^{+1} \Gamma(x, y; \lambda)h(y)dy + A\xi_1(x) + B\xi_2(x) \tag{8.1}$$

by the theorems of §§4 and 7. The ξ_i are defined in (7.2). We desire that f be a bounded linear transformation of h. Hence A and B must be linear functionals on h, which we will denote by $Q_1^*(h)$ and $Q_2^*(h)$, respectively. The resolvent then has the form

$$f(x) = \int_{-1}^{+1} \Gamma(x, y; \lambda)h(y)dy + Q_1^*(h)\xi_1(x) + Q_2^*(h)\xi_2(x). \tag{8.2}$$

The most general lateral condition will, of course, involve both boundaries. *In order to simplify notation we shall assume, until* §14, *the left boundary condition* $f(-1) = 0$. When the result is obtained for this case, the passage to two free boundaries is exactly the same as for the parabolic partial differential equations. *Since the Hille-Yosida theorem requires that the domain of the infinitesimal generator be dense in the underlying Banach space, we take as the space* X *of Theorem* 1.1, *the space* $C_0[-1, +1]$ *of the following definition.* This does not mean that the semi-group cannot be extended to $C[-1, +1]$; it means that the solutions of (0.1) will converge *uniformly* to the initial distributions only within the subspace C_0.

DEFINITION 8.1. *We define* $C_0[-1, +1]$ *to be the space of all functions which are continuous on* $[-1, +1]$ *and which vanish at* -1.

In this space the resolvent transformation is given by

$$f(x) = \int_{-1}^{+1} \Gamma(x, y; \lambda)h(y)dy + Q^*(h)\xi_2(x) \tag{8.3}$$

where $Q^*(h)$ is a linear functional.

We shall now extend the results of §4.

THEOREM 8.1. *The transformation* (8.3) *is positivity preserving and* $\lambda\|f\| \leqq \|h\|$ *if and only if* Q^* *is a positive functional with* $\|Q^*\| \leqq 1$.

Proof. First suppose that (8.3) is positivity preserving and $\lambda\|f\| \leqq \|h\|$. If $h \geqq 0$, then letting $x \to 1$ in (8.3) we get $Q^*(h) \geqq 0$ since $\lambda > 0$. Also $Q^*(h) = \lambda \cdot f(1) \leqq \|h\|$.

Now suppose that $Q^* \geqq 0$ and $\|Q^*\| \leqq 1$. Since $\xi_2(x) \geqq 0$ by Theorem 7.2, the transformation (8.3) is positivity preserving. Also, by (7.8)

$$\lambda\|f\| \leqq \|h\| \left[\lambda\xi_2(x)(\|Q^*\| - 1) - \lambda\xi_1(x) + 1\right]. \tag{8.4}$$

Since $\|Q^*\| \leqq 1$ and by (7.8) also $0 \leqq \lambda^{-1} - \xi_1 - \xi_2 \leqq \lambda^{-1}$, the quantity in brackets is <1. This completes the proof.

LEMMA 8.1. *If* $h \in C_0[-1, +1]$ *and* $f(x)$ *is of the form* (8.3), *then*

$$\begin{aligned} &\pi 2^{1/2} \lim_{\epsilon\to 0} \epsilon^{-1} \int_{1-\epsilon}^{1} f'(x)(1-x)^{1/2}dx \\ &\qquad = \delta_2 f = -\int_{-1}^{+1} h(y)\xi_2^*(y)dy + Q^*(h)\delta_2\xi_2 \end{aligned} \tag{8.5}$$

with ξ_2^* *defined in* (4.7).

Proof. The existence of $\delta_2 f$ and the equation on the right of (8.5) are direct consequences of Theorem 4.3 and the definition of ξ_2 in (7.7). To show that the left equation holds in general, we write

$$\begin{aligned} \mu(\epsilon) &= \epsilon^{-1} \int_{1-\epsilon}^{1} f'(x)(1-x)^{1/2}dx \\ &= \frac{f(1) - f(1-\epsilon)}{\epsilon^{1/2}} - \frac{1}{2\epsilon}\int_{1-\epsilon}^{1} \frac{f(1) - f(x)}{(1-x)^{1/2}}dx = \psi'(\epsilon) - \frac{1}{2\epsilon}\psi(\epsilon). \end{aligned} \tag{8.6}$$

If $\delta_2 f$ exists, it is clear that $\pi 2^{1/2} \lim_{\epsilon\to 0} \mu(\epsilon) = \delta_2 f$. Conversely, if $\lim_{\epsilon\to 0} \mu(\epsilon)$ exists, we solve the differential equation in (8.6) to get $\epsilon^{-1/2}\psi(\epsilon) = \int_0^\epsilon \mu(x)x^{-1/2}dx_1$ since we know that $\lim_{\epsilon\to 0} \epsilon^{-1/2}\psi(\epsilon) = 0$. Hence $\lim_{\epsilon\to 0} \epsilon^{-1}\psi(\epsilon)$ exists, which implies the existence of $\delta_2 f$.

9. The lateral condition for one free boundary.

DEFINITION 9.1. *For given constants* $\tau \geqq 0$, $\sigma \geqq 0$, *and* $p_2 \geqq 0$ *such that*

$$\tau \leqq p_2 \tag{9.1}$$

and a given nondecreasing function $p(x)$ *such that*

$$\lim_{x\to -1+} p(x) = 0, \qquad \lim_{x\to +1^-} p(x) = 1, \tag{9.2}$$

we define Σ *as the set of all* $f \in C_0[-1, +1]$ *for which*

$$p_2 f(1) = \tau \int_{-1}^{+1} f(x)dp(x) - \sigma \lim_{x\to 1} \Omega f(x) - \gamma\delta_2 f \tag{9.3}$$

where $\delta_2 f$ is defined in (4.5).

THEOREM 9.1. *The operator Ω of* (1.3), *restricted by the lateral condition* (9.3), *is the infinitesimal generator of a contraction semi-group with range dense in $C_0[-1, +1]$ or in the linear manifold defined by*

$$p_2 f(1) = \tau \int_{-1}^{+1} f(x)dp(x) \tag{9.4}$$

according as $\sigma+\gamma>0$ or $\sigma+\gamma=0$. Its resolvent is given by the linear transformation

$$f(x) = \int_{-1}^{+1} \Gamma(x, y; \lambda)h(y)dy + \xi_2(x)\left[q_2\cdot h(1) + \int_{-1}^{+1} h(y)\cdot Q(y)\cdot dy\right] \tag{9.5}$$

with

$$q_2 = \sigma\cdot A, \tag{9.6}$$

$$Q(y) = \tau A\int_{-1}^{+1} \Gamma(x, y; \lambda)dp(x) + \gamma A\xi_2^*(y), \tag{9.7}$$

where ξ_2^* is defined in (4.7) *and*

$$A = \frac{\lambda}{p_2 - \lambda\tau\int_{-1}^{+1}\xi_2(x)dp(x) + (\sigma+\gamma\omega)\lambda} \tag{9.8}$$

with $\omega=\delta_2\xi_2$ as defined in (4.5).

The next section will be devoted to the proof of this theorem.

10. Proof of Theorem 9.1.

LEMMA 10.1. *Given*

$$f \in D(\Omega) = \textit{domain } \Omega,$$

define h by (2.1). *Condition* (9.3) *is equivalent to*

$$f(1) = \lambda\left[q_2 h(1) + \int_{-1}^{+1} Q(s)h(s)ds\right] \tag{10.1}$$

where q_2 and $Q(s)$ are defined in (9.6) *and* (9.7).

Proof. Suppose first that $f\in\Sigma$. Then clearly, since by Theorem 7.1 the only solution of (7.1) in $C_0[-1, +1]$ is $\xi_2(x)$, we can write

$$f(x) = \int_{-1}^{+1} \Gamma(x, y)h(y)dy + \lambda\xi_2(x)f(1). \tag{10.2}$$

If $f \in \Sigma$, then from (10.2), (9.3), and Lemma 8.1

$$\begin{aligned} p_2 f(1) = {} & \tau \int_{-1}^{+1} \left[\int_{-1}^{+1} \Gamma(x, y)h(y)dy \right] dp(x) \\ & + \lambda f(1) \int_{-1}^{+1} \xi_2(x)dp(x) - \sigma \lim_{x\to 1} [\lambda f(x) - h(x)] \\ & + \gamma \int_{-1}^{+1} \xi_2^*(x)h(x)dx - \lambda f(1)\gamma\omega. \end{aligned} \tag{10.3}$$

A rearrangement of the terms in (10.3) gives (10.1). The steps can be reversed to show that (10.1) implies (9.3).

Lemma 10.2. *A function $f \in D(\Omega)$ is representable in the form* (9.5) *if and only if $f \in \Sigma$.*

Proof. Again define h as in (2.1). We can express f in the form (10.2) by the argument used in Lemma 10.1. Hence, a necessary and sufficient condition that f be in the range of (9.5) is that (10.1) be satisfied. By Lemma 10.1 this is equivalent to (9.3).

Lemma 10.3. *The functional $Q^*(h) = q_2 h(1) + \int_{-1}^{+1} Q(s)h(s)ds$ where q_2 and $Q(s)$ are defined in* (9.6) *and* (9.7) *is a positive functional of norm less than one.*

Proof. We have

$$\begin{aligned} \|Q^*\| &= q_2 + \int_{-1}^{+1} Q(s)ds \\ &= A\left[\sigma + \tau \int_{-1}^{+1} \left\{ \int_{-1}^{+1} \Gamma(x, y)dy \right\} dp(x) + \gamma \int_{-1}^{+1} \xi_2^*(x)dx \right]. \end{aligned} \tag{10.4}$$

It is easily seen from (7.7) and Theorem 4.3 that

$$\omega = \int_{-1}^{+1} \xi_2^*(x) \cdot \eta_2(x)dx + \lambda^{-1} \tag{10.5}$$

where ω is given in (9.8). Hence the right-hand side of (10.4) is equal to

$$\frac{\lambda(\sigma+\gamma\omega) + \tau - \lambda\tau \int_{-1}^{+1} (\xi_1+\xi_2)dp(x) - \gamma\left[1 - \lambda \int_{-1}^{+1} \xi_2^*(1-\eta_2)dx\right]}{\lambda(\sigma + \gamma\omega) + p_2 - \tau\lambda \int_{-1}^{+1} \xi_2 dp(x)}. \tag{10.6}$$

But

$$1-\lambda\int_{-1}^{+1}\xi_2^*(x)[1-\eta_2(x)]dx = 1-\lambda\int_{-1}^{+1}\xi_2^*(x)\eta_1(x)dx \tag{10.7}$$
$$= -\delta_2\xi_1\lambda \geqq 0,$$

since $\xi_1 \geqq 0$ and $\xi_1(1)=0$. From (9.1) and (10.7) we conclude that $\|Q^*\| \leqq 1$.

The next three lemmas are used only for the statement concerning the density of Σ.

LEMMA 10.4. *Given $\epsilon>0$, there is a $w(x)\in D(\Omega)$ such that $\|w\|<\epsilon$ and*

$$|\delta_2 w| < \epsilon, \qquad 1 = \lim_{x\to 1}\Omega w(x). \tag{10.8}$$

Proof. Let $h(x)$ be the function defined by

$$h(x) = \begin{cases} 0 & (-1 \leqq x \leqq 1-\delta), \\ \delta^{-1}x + 1 - \delta^{-1} & (1-\delta < x \leqq 1). \end{cases} \tag{10.9}$$

Define $w(x)=\int_{-1}^{+1}K(x, y)h(y)dy$. Then

$$\|w\| \leqq \left\|\int_{1-\delta}^{+1} K(x, y)dy\right\| < \epsilon \tag{10.10}$$

for δ sufficiently small by Lemma 5.3. Also by the argument used in Theorem 4.3

$$\delta_2 w = \pi^{-1}\int_{-1}^{+1}(1+x)^{1/2}(1-x)^{-1/2}h(x)dx < \epsilon \tag{10.11}$$

for δ sufficiently small.

LEMMA 10.5. *Given $\epsilon>0$ there exists a $v(x)\in D(\Omega)$ such that $\|v\|<\epsilon$ and*

$$\lim_{x\to 1}(1-x^2)^{1/2}v'(x) = 1. \tag{10.12}$$

Proof. Let

$$v_n(x) = \int_{-1}^{x}\frac{t^{2n}}{(1-t^2)^{1/2}}dt. \tag{10.13}$$

We can choose n large enough so that $\|v_n\|<\epsilon$. Furthermore $\Omega v_n(x)$ is a polynomial by (1.5), so that $v_n \in D(\Omega)$.

LEMMA 10.6. *The set Σ defined in Definition* 9.1 *is dense in* $C_0[-1, +1]$ *if $\sigma+\gamma>0$ and is dense in the manifold defined by* (9.4) *if $\sigma+\gamma=0$.*

Proof. First suppose $\sigma>0$. Since $D(\Omega)$ is dense in $C_0[-1, +1]$ it will be sufficient to show that to each $h\in D(\Omega)$ and each $\epsilon>0$, we can find a w_ϵ such that $\|w_\epsilon\| < \epsilon$ and $h + w_\epsilon \in \Sigma$. This can be done by choosing a constant A so that

$$(10.14)\qquad \begin{aligned} &p_2 h(1) - \int_{-1}^{+1} h(x)dp(x) + \gamma\delta_2 h + \sigma \lim_{x\to 1} \Omega h(x) - A\epsilon(w) - A\sigma \\ &\qquad = h_0 - A\epsilon(w) - A\sigma = 0, \end{aligned}$$

where w satisfies (10.8) and

$$(10.15)\qquad \epsilon(w) = p_2 w(1) - \tau \int_{-1}^{+1} w(x)dp(x) + \gamma\delta_2 w.$$

This gives $A = h_0(\epsilon+\sigma)^{-1}$ and since, by Lemma 10.4, we can choose w so that $\epsilon(w)$ is arbitrarily small, there are finite values of A satisfying (10.14). We can now put $w_\epsilon = Aw$ with $\|Aw\| < \epsilon$. If $\sigma = 0$ but $\gamma \neq 0$, then the proof follows along the same lines using Lemma 10.5. The statement concerning the case $\sigma + \gamma = 0$ is obvious.

Proof of the theorem. Suppose a set Σ of the type in Definition 9.1 is given. The function $f(x)$ defined by (9.5), (9.6), (9.7), and (9.8) is a solution of $\lambda f - \Omega f = h$ by Theorems 4.2 and 7.1. Lemmas 10.1 and 10.2 show that, given $h \in C_0[-1, +1]$ there is a unique solution of $\lambda f - \Omega f = h$ in Σ. Lemma 10.3 and Theorem 8.1 show that f is positivity preserving and $\lambda\|f\| \leqq \|h\|$. These results combined with Lemma 10.6 show that Ω/Σ satisfies the conditions of the Hille-Yosida Theorem (Theorem 1.1), which proves Theorem 9.1.

11. **The adjoint semi-groups.** We now wish to investigate the semi-groups adjoint to those generated by Ω. This requires a study of the adjoint resolvent. We introduce the $\bar{L}_0$ following [3].

DEFINITION 11.1. *We denote by $\bar{L}_0[-1, +1]$ the Banach space of pairs $[g(x), g_2]$ where $g(x) \in L[-1, +1]$ and g_2 is a real number, and* $\|[g(x), g_2]\| = |g_2| + \|g\|_L$.

Although the adjoint resolvent is a transformation from B.V. $[-1, +1]$ to itself, we shall not need the full adjoint for our purposes. As in the differential equations case our adjoint resolvent maps the Banach space B.V. $[-1, +1]$ into the space $\bar{L}_0[-1, +1]$ and we must, therefore, in order to use the Hille-Yosida Theorem, consider the contraction of the adjoint to $\bar{L}_0[-1, +1]$. In the case of two free boundaries we must take the space $\bar{L}[-1, +1]$ of *triples* $[g_1, g(x), g_2]$ with the obvious norm.

Let $[g(x), g_2]$ be an arbitrary element of $\bar{L}_0$. Let $F \in C_0$ and $f = R(\lambda)\cdot F$, and $G = R^*(\lambda)g$. We then have

$$(11.1)\qquad g_2 f(1) + \int_{-1}^{+1} f(x)g(x)dx = G_2 F(1) + \int_{-1}^{+1} F(x)G(x)dx.$$

Now, substituting (9.5) into (11.1) we obtain

$$G_2 = g_2 \cdot q_2 \lambda^{-1} + q_2 \int_{-1}^{+1} \xi_2(x) g(x) dx, \tag{11.2}$$

$$G(s) = Q(s) \left\{ g_2 \lambda^{-1} + \int_{-1}^{+1} \xi_2(x) g(x) dx \right\} + \int_{-1}^{+1} \Gamma(x, s) g(x) dx. \tag{11.3}$$

In this section we prove some preparatory lemmas.

LEMMA 11.1. *Let* $v(x) \in$ B.V. $[-1, +1]$ *be monotonic with* $\lim_{x\to -1^+} v(x) = v_1$ *and* $\lim_{x\to +1^-} v(x) = v_2$. *Define*

$$\bar{v}(x) = \int_{-1}^{+1} \Gamma(x, y) dv(y), \tag{11.4}$$

then $\bar{v} \in L[-1, +1]$ *and*

$$\pi^{-1} \operatorname*{essential\,limit}_{x \to (-1)^j} P \int_{-1}^{+1} \frac{\bar{v}(t) \cdot dt}{t - x} = \Phi_j \bar{v}, \qquad j = 1, 2, \tag{11.5}$$

exists. Furthermore, $\bar{v}$ *satisfies*

$$\lambda \int_{x_1}^{x_2} \bar{v}(x) dx - \left\{ \pi^{-1} P \int_{-1}^{+1} \frac{v(t) dt}{t - x} \right\}_{x_1}^{x_2} = v(x_2) - v(x_1) \tag{11.6}$$

for each pair $x_1, x_2 \in [-1, +1]$.

Proof. It is easy to show, using (2.9), that $\bar{v}(x)$ is a solution of

$$\bar{v}(x) + \lambda \int_{-1}^{+1} K(x, y) \bar{v}(y) dy = \int_{-1}^{+1} K(x, y) dv(y). \tag{11.7}$$

A formal integration by parts in (11.7) gives

$$\bar{v}(x) = (1 - x^2)^{1/2} (\pi^{-1}) P \int_{-1}^{+1} \frac{\lambda \bar{V}(y) + v(y)}{(1 - y^2)^{1/2} (y - x)} dy \qquad \text{a.e.} \tag{11.8}$$

where $\bar{V}(x) = \int_x^1 \bar{v}(x) dx$. To justify this, first note that the integrals in (11.7) are finite for almost all x. For such a value of x, we can integrate by parts over the region $-1 < x - \epsilon < y < x + \epsilon < 1$. It is easily seen from (5.10) that $\lim_{\epsilon \to 0} [K(x, x+\epsilon) - K(x, x-\epsilon)] = 0$ for all x and that $\lim_{\epsilon \to 0} K(x, x \pm \epsilon) \cdot [v(x+\epsilon) - v(x-\epsilon)] = 0$ for almost all x.

It has been shown by Ahiezer [1, cf. Lemma 1.2] that

$$Tf = P \int_{-1}^{+1} \frac{f(y) dy}{(1 - y^2)^{1/2} (y - x)}$$

is a transformation from the space $L_2[(1 - x^2)^{-1/2}]$ of functions square integrable on $[-1, +1]$ with respect to the weight function $(1 - x^2)^{-1/2}$ to the space $L_2[(1 - x^2)^{1/2}]$. Since $\lambda \bar{V} + v$ is an element of the former space, it follows that $\bar{v}^2(1-x^2)^{-1/2}\in L[-1, +1]$(4). We may now apply Lemma 2.1 to obtain

$$(11.9)\qquad \lambda\tilde{V}(x) + v(x) = -\pi^{-1}P\int_{-1}^{+1}\frac{\bar{v}(t)}{t-x}dt + A \qquad \text{a.e.}$$

This proves the existence of the Φ_j. Since $\lim_{x\to 1}(\lambda\tilde{V}+v)=v_2$, we must put $A=\text{ess}\lim_{x\to 1}\pi^{-1}P\cdot\int_{-1}^{+1}(\bar{v}(t)/(t-x))dt+v_2$. Clearly (11.6) follows from (11.9).

COROLLARY 11.1. *If $v(x)$ is absolutely continuous in $[-1, +1]$, then $\bar{v}\in D(\Omega^*)$ and*

$$(11.10)\qquad \lambda\bar{v} - \Omega^*\bar{v} = v'.$$

NOTATIONAL CONVENTION. *Following the convention used in* [3] *for the differential equations case, we shall write* (11.6) *as*

$$(11.11)\qquad \lambda\bar{v} - \Omega^*\bar{v} \cong \Delta v.$$

In case v is absolutely continuous we replace Δv by v' or use (11.10), *which is equivalent to* (11.11) *for absolutely continuous v.*

LEMMA 11.2. *In the notation of Lemma* 11.1

$$(11.12)\qquad \lambda\int_{-1}^{+1}\bar{v}(x)\eta_2(x)dx + \pi^{-2}\int_{-1}^{+1}(1-t^2)^{-1/2}\left[P\int_{-1}^{+1}\frac{\bar{v}(x)dx}{x-t}\right]dt - \Phi_2\bar{v} = \int_{-1}^{+1}\eta_2(x)dv(x).$$

Proof. Using Lemma 11.1, we have

$$(11.13)\qquad \lambda\int_{-1}^{+1}\tilde{V}(x)\eta_2'(x)dx + \pi^{-2}\int_{-1}^{+1}\eta_2'(x)\left\{\overline{P}\int_{-1}^{+1}\frac{\bar{v}(t)dt}{t-x}\right\}dx = -\int_{-1}^{+1}v(x)\eta_2'(x)dx + v_2 + \Phi_2$$

where $\tilde{V}(x)=\int_x^1 v(x)dx$. An integration by parts gives the result.

DEFINITION 11.2. *We define*

$$(11.14)\qquad \begin{aligned} L_1\bar{v} &= \lim_{\epsilon\to 0}\frac{1}{2(2\epsilon)^{1/2}}\int_{-1}^{-1+\epsilon}\bar{v}(x)dx,\\ L_2\bar{v} &= \lim_{\epsilon\to 0}\frac{1}{2(2\epsilon)^{1/2}}\int_{1-\epsilon}^{1}\bar{v}(x)dx.\end{aligned}$$

LEMMA 11.3. *If $v\in$ B.V. $[-1, +1]$ and $\bar{v}$ is defined as in* (11.4), *then $L_i\bar{v}=0$ for $i=1, 2$.*

(4) It is also quite easy to verify this directly.

Proof. Since from

$$(11.15)\qquad \int_{-1}^{+1}\left[\int_{1-\epsilon}^{1}\Gamma(x, y)dy\right]|dv(x)| \leqq \int_{-1}^{+1}\left[\int_{1-\epsilon}^{1}K(x, y)dy\right]|dv(x)|,$$

it will be sufficient to show that the last integral is $o(\epsilon^{1/2})$. Without loss of generality, we may assume that $v(x)$ is nondecreasing. We introduce the following notations for this proof:

$$(11.16)\qquad \begin{aligned} I_1 &= \int_{-1}^{1-2\epsilon}\left\{\int_{1-\epsilon}^{1}K(x, y)dy\right\}dv(x), \\ I_2 &= \int_{1-2\epsilon}^{1}\left\{\int_{1-\epsilon}^{1}K(x, y)dy\right\}dv(x) \end{aligned}$$

and

$$(11.17)\qquad K_i(x, y) = \log\left(1 - x\cdot y - (-1)^i((1 - x^2)(1 - y^2))^{1/2}\right), \qquad i = 1, 2.$$

Note that $|x-y| > \epsilon$ throughout the region of integration in I_1. In this region for ϵ sufficiently small, using (5.10),

$$(11.18)\qquad K(x, y) = |K_1| + |K_2| \leqq 2\log \epsilon^2/2$$

and so

$$(11.19)\qquad |I_1| \leqq [2\epsilon|\log \epsilon^2/2|]\operatorname{var} v.$$

We next estimate I_2. Performing the inner integration we get

$$(11.20)\qquad |I_2| \leqq \int_{1-2\epsilon}^{1}|x - 1 + \epsilon|K(x, 1 - \epsilon)dv(x) + o(\epsilon^{1/2}).$$

In the region of integration, for ϵ small enough, $K(x, \epsilon) \leqq 4\log(|x-1+\epsilon|/2)$, from (11.20) and the definition of I_2. Hence $I_2 = o(\epsilon^{1/2})$, and $L_2 = 0$. Because of the symmetry of $\Gamma(x, y)$, the proof that $L_1 = 0$ is the same.

LEMMA 11.4. *The general solution of*

$$(11.21)\qquad \lambda f - \Omega^* f = 0$$

in $L[-1, +1]$ *is given by* $f = A\xi_1^* + B\xi_2^*$ *where* A *and* B *are constants and* ξ_j^* *is defined in Theorem* 4.3.

Proof. That ξ_1^* and ξ_2^* are solutions of (11.21) follows from (1.5) and Theorem 4.2. Suppose f is a third solution. By the duality principle stated in §1, there exist constants a, b, and c such that $\int_a^x f$, $\int_b^x \xi_1^*$, and $\int_c^x \xi_2^*$ are solutions of $\lambda F - \Omega F = 0$. By Theorem 7, there exist constants A, B, and C such that $C\int_a^x f = A\int_b^x \xi_1^* + B\int_c^x \xi_2^*$. Since ξ_1^* and ξ_2^* are independent, $C \neq 0$, and the lemma is proved.

COROLLARY 11.2. *No solution of* (11.21) *in* $L[-1, +1]$ *satisfies* $L_1 f = L_2 f = 0$.

Proof. This follows from Lemma 11.4 and the fact that

$$L_i \xi_j^* = \begin{cases} 1 & (i = j), \\ 0 & (i \neq j). \end{cases} \tag{11.22}$$

LEMMA 11.5. *If* $\int_{-1}^{+1} v^2 (1-x^2)^{-1/2} dx < \infty$, *then*

$$\int_{-1}^{+1} (1 - t^2)^{-1/2} \left[P \int_{-1}^{+1} \frac{v(x)dx}{x - t} \right] dt = 0. \tag{11.23}$$

Proof. It is not difficult to verify this by direct methods, but a more instructive proof is the following. Letting $\phi(x) = v(x)(1-x^2)^{-1/2}$, we have $\int_{-1}^{+1} \phi^2(x)(1-x^2)^{1/2} dx < \infty$. Using Lemma 1.2, we put $\phi(x) = \sum_{k=1}^{\infty} a_k \cdot P_k(x)$ and conclude that $T\phi = \sum_{k=1}^{\infty} a_k Q_{k+1}(x)$. Hence (11.23) is equal to $\sum_{k=1}^{\infty} a_k \int_{-1}^{+1} Q_{k+1}(x)(1-x^2)^{-1/2} dx = 0$.

12. The adjoint resolvent and the forward equation.

THEOREM 12.1. *If* $\sigma > 0$, *then* (11.2) *and* (11.3) *define the resolvent of a contraction semi-group with range dense in* $\overline{L}_0[-1, +1]$. *The resolvent can be written*

$$[G(x), G_2] = (\lambda I - \overline{\Omega}^*)^{-1} [g(x), g_2] \tag{12.1}$$

where $\overline{\Omega}^*$ *is defined by*

$$\lambda G - \Omega^* G - \frac{\tau}{\sigma} \Delta p \cong g, \tag{12.2}$$

$$\lambda G_2 + \frac{p_2}{\sigma} \cdot G_2 + \Phi_2 G = g_2, \tag{12.3}$$

where Φ_2 *is defined in* (11.5), *restricted by the lateral condition*

$$\sigma L_2 G = \gamma G_2, \qquad L_1 G = 0 \tag{12.4}$$

with L_i *defined as in* (11.14).

If $\sigma = 0$, *then* $G_2 = 0$ *and we restrict considerations to the contraction of the adjoint resolvent to* $L[-1, +1]$. *This contraction has range dense in* $L[-1, +1]$ *and is the inverse of* $\lambda I - \overline{\Omega}^*$, *where* $\overline{\Omega}^*$ *is defined by*

$$\lambda G - \Omega^* G - \frac{\tau}{p_2} \Phi_2 \cdot \Delta p \cong g \tag{12.5}$$

restricted by the lateral condition

$$L_2 G = - \frac{\gamma}{p_2} \Phi_2 G, \qquad L_1 G = 0. \tag{12.6}$$

The next section will be devoted to the proof of this theorem. We wish to remark that if $\tau=0$ then (12.5) becomes $\lambda G-\Omega^*G=0$ and (12.6) is the analogue of the classical boundary conditions for the forward equation of diffusion theory.

13. **Proof of Theorem 12.1.**

LEMMA 13.1. *The range of the transformation on* $\overline{L}_0[-1, +1]$ *defined by* (11.2) *and* (11.3) *is independent of* λ *and dense in* $\overline{L}_0[-1, +1]$ *or* $L[-1, +1]$ *according as* $\sigma>0$ *or* $\sigma=0$.

Proof. For fixed λ a function G is representable in the form (11.3) if and only if it is of the form

$$G = A\tilde{G} + BH \tag{13.1}$$

where A and B are constants and

$$\lambda\tilde{G} - \Omega^*\tilde{G} = g, \qquad \lambda H - \Omega^* H \cong \Delta p \tag{13.2}$$

for some $g \in L[-1, +1]$ and some p of the form described in Definition 9.1. The set defined by (13.1) and (13.2) is independent of λ, since if (13.2) holds for $\lambda=\lambda_0$, then

$$\begin{aligned} \lambda\tilde{G} - \Omega^*\tilde{G} &= g + (\lambda-\lambda_0)G, \\ \lambda H - \Omega^* H &\cong \Delta p + (\lambda-\lambda_0)H \end{aligned} \tag{13.3}$$

and so G is of the form (13.1).

The density of the range follows from the fact that $D(\Omega^*)$ is dense in $L[-1, +1]$ and that, given $g\in L[-1, +1]$, we can determine g_2 so that (11.2) is satisfied.

LEMMA 13.2. *The transformation from* $\overline{L}_0[-1, +1]$ *to itself, defined by* (11.2) *and* (11.3), *is positivity preserving and*

$$\lambda\| [G, G_2]\| < \| [g, g_2]\|. \tag{13.4}$$

Proof. The positivity is clear. To prove the norm condition we need consider only the case $g\geqq 0$. We have, using (7.8),

$$\begin{aligned} \lambda\| [G, G_2]\| = {} & \left\{g_2 + \lambda\int_{-1}^{+1} \xi_2(x)g(x)dx\right\}\left\{q_2 + \int_{-1}^{+1} Q(s)ds\right\} \\ & + \int_{-1}^{+1} g(s)\{1-\lambda\xi_1(s)-\lambda\xi_2(s)\}ds, \end{aligned} \tag{13.5}$$

and since $\|Q^*\|\leqq 1$ from Lemma 10.3, it follows that

$$\lambda\| [G, G_2]\| \leqq g_2 + \int_{-1}^{+1} g(x)dx - \lambda\int_{-1}^{+1}\xi_1(x)g(x)dx < \| [g, g_2]\|. \tag{13.6}$$

Proof of the theorem. (1) *The case* $\sigma>0$. Equation (12.2) is an obvious consequence of Lemma 11.1 and Corollary 11.1.

To prove (12.3), let $g(x)$ be the function defined in (11.10). By Corollary 11.1, we have

$$\lambda\int_{-1}^{+1}\bar{g}(x)\eta_2(x)dx-\int_{-1}^{+1}g(x)\eta_2(x)dx=\int_{-1}^{+1}\eta_2(x)\left[\Omega^*g(x)\right]dx. \tag{13.7}$$

On the other hand, from the definition of g and (7.7) we get

$$\lambda\int_{-1}^{+1}\bar{g}(x)\eta_2(x)dx=\int_{-1}^{+1}g(x)\left[\eta_2(x)-\lambda\xi_2(x)\right]dx, \tag{13.8}$$

and therefore

$$\lambda\int_{-1}^{+1}g(x)\xi_2(x)dx=-\int_{-1}^{+1}\eta_2(x)\Omega^*g(x)dx=-\Phi_2g+T_2g \tag{13.9}$$

where

$$T_2g=\pi^{-2}\int_{-1}^{+1}(1-t^2)^{-1/2}\left[\overline{P}\int_{-1}^{+1}\frac{g(x)dx}{x-t}\right]dt. \tag{13.10}$$

In a similar way, using Lemmas 11.1 and 11.2, we have

$$\lambda\tau A\int_{-1}^{+1}\xi_2(x)dp(x)-\lambda\gamma A\int_{-1}^{+1}\xi_2^*(x)\eta_2(x)dx=-\Phi_2Q+T_2Q. \tag{13.11}$$

Hence from (10.5)

$$\lambda\tau A\int_{-1}^{+1}\xi_2(x)dp(x)-\lambda\gamma A\omega+\gamma A=-\Phi_2Q+T_2Q. \tag{13.12}$$

From (13.9) and (13.12)

$$\begin{aligned}&-\Phi_2G+T_2G\\ &=\left\{g_2+\lambda\int_{-1}^{+1}\xi_2(x)g(x)dx\right\}\left\{\tau A\int_{-1}^{+1}\xi_2(x)dp(x)-\gamma A\omega+1\right\}\\ &\quad-g_2+\frac{\gamma}{\sigma}G_2=-g_2+\lambda G_2+\frac{p_2+\gamma}{\sigma}G_2.\end{aligned} \tag{13.13}$$

From (1.5) we have

$$T_2\left(\frac{1+x}{1-x}\right)^{1/2}=1,$$

hence, by Lemma 11.5, $T_2G=(\gamma/\sigma)G_2$. This substitution in (13.13) gives (12.3).

That the lateral condition (12.4) holds follows from Lemma 11.3 and the definition of G.

If the solution were not unique, we could find a solution of (11.21) which satisfies

$$L_1 f = L_2 f = 0,$$

but this contradicts Corollary 11.2.

(2) *The case* $\sigma = 0$. In this case we may, for our purposes, consider the adjoint resolvent as a transformation from $L[-1, +1]$ to itself since $G_2 = 0$. This contraction is given by

$$G(x) = g(x) + Q(x) \int_{-1}^{+1} \xi_2(s) g(s) ds. \tag{13.14}$$

By a calculation similar to that used in the first equation of (13.13) we have

$$A \int_{-1}^{+1} \xi_2(x) g(x) dx = (T_2 G - \Phi_2 G)(p_2 + \gamma)^{-1}. \tag{13.15}$$

Also by Lemmas 11.5 and 11.3 and (11.22)

$$T_2 G = \gamma A \int_{-1}^{+1} \xi_2(x) g(x) dx = L_2 G. \tag{13.16}$$

Hence,

$$(p_2 + \gamma) L_2 G = \gamma (T_2 G - \Phi_2 G). \tag{13.17}$$

By (13.15) and (13.16) this reduces to (12.6). Equation (12.5) now follows from this result.

14. The two boundary case. In the previous sections we have assumed for simplicity the boundary condition $f(-1) = 0$ and have shown that the semi-groups generated by Ω and their adjoints have the same structure as the semi-groups associated with the parabolic differential equations with "regular boundaries," cf. [3]. It is only the analytic method of establishing this structure which is different in our case. Once this analogy has been proved, the method given in [3] for the transition from one free boundary to two is applicable. A comparison of our resolvents and lateral conditions to those given in [3, §19] shows that they have the same form with $\lim_{\epsilon \to 0} (f(1) - f(1-\epsilon)/\epsilon^{1/2}$ replacing $\lim_{x \to 1} e^{B(x)} f'(x)$ for the backward equation, and $\lim_{x \to 1} (G'(x) - b(x)G(x))$ replaced by ess. $\lim_{x \to 1} \pi^{-1} P \int_{-1}^{+1} G(x) dx/(x-t)$ and $\lim_{x \to 1} e^{-B(x)} G(x)$ by $\lim_{\epsilon \to 0} (2(2\epsilon)^{1/2})^{-1/2} \int_{1-\epsilon}^{1} G(x) dx$ in the resolvents and the lateral conditions of the forward equation. As was pointed out in the introduction, this means that the analogue to the classical boundary conditions is (0.9–10). In view of this, we shall only state the lateral conditions here and refer the reader to [3] for the proofs and for the form of the resolvents. It is necessary to introduce a matrix notation for the expression of the resolvents.

The conditioning set Σ for the backward equation is the set of functions such that

$$
(14.1)\quad
\begin{aligned}
(1-\pi_1-p_{11})f(-1)-p_{12}f(1) &= \tau_1\int_{-1}^{+1} f(x)dp_1(x)-\sigma_1\lim_{x\to -1}\Omega f(x)\\
&\qquad +\pi_1\lim_{\epsilon\to 0}\frac{\pi}{2^{1/2}}\,\frac{f(-1)-f(-1+\epsilon)}{\epsilon^{1/2}},\\
-p_{21}f(-1)+(1-\pi_2-p_{22})f(1) &= \tau_2\int_{-1}^{+1} f(x)dp_2(x)-\sigma_2\lim_{x\to 1}\Omega f(x)\\
&\qquad -\pi_2\lim_{\epsilon\to 0}\frac{\pi}{2^{1/2}}\,\frac{f(1)-f(1-\epsilon)}{\epsilon^{1/2}}
\end{aligned}
$$

where

$$
(14.2)\quad
\begin{aligned}
&p_{ij}\geqq 0,\qquad \pi_i\geqq 0,\qquad \sigma_i\geqq 0,\qquad \tau_i\geqq 0,\\
&p_{i1}+p_{i2}+\pi_i+\tau_i\leqq 1, \qquad\qquad i=1,2,
\end{aligned}
$$

and $p_i(x)$ are nondecreasing, with

$$
(14.3)\quad \lim_{x\to -1^+} p_j(x)=0,\qquad \lim_{x\to +1^-} p_j(x)=1.
$$

We may replace the last term in each of the lateral conditions in (14.1) by the equivalent $\lim_{\epsilon\to 0}\pi\epsilon^{-1}\int_{-1}^{-1+\epsilon}f'(x)(1-x^2)^{1/2}dx$ and $\lim_{\epsilon\to 0}\pi\epsilon^{-1}\int_{1-\epsilon}^{1}f'(x)(1-x^2)^{1/2}dx$, respectively. Conditions (14.2) are necessary for the positivity preservation.

As in the differential equations case the relatively complicated lateral conditions for the backward equation all lead to the following simple conditions for the forward equation

$$
(14.4)\quad
\begin{aligned}
\pi_1\cdot G_1 &= \sigma_1\lim_{\epsilon\to 0}(2(2\epsilon)^{1/2})^{-1}\int_{-1}^{-1+\epsilon}G(x)dx,\\
\pi_2\cdot G_2 &= \sigma_2\lim_{\epsilon\to 0}(2(2\epsilon)^{1/2})^{-1}\int_{1-\epsilon}^{1}G(x)dx
\end{aligned}
$$

where $\sigma_i\neq 0$. If $\sigma_1=0$ we get the analogue of the classical boundary condition, namely $(0\cdot 9-10)$.

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Mount Holyoke College,
South Hadley, Mass.